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LETTER TO THE EDITOR

Galilean covariance and the Duffin–Kemmer–Petiau equation

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Abstract. We use a five-dimensional approach to Galilean covariance to investigate the non-relativistic Duffin–Kemmer–Petiau first-order wave equations for spinless particles. The corresponding representation is generated by five 6×6 matrices. We consider the harmonic oscillator as an example.

The purpose of this letter is to apply Galilei covariance in a five-dimensional approach to construct Duffin–Kemmer–Petiau (DKP) first-order wave equations. After illustrating our formalism by first recovering the Schrödinger equation, we then determine the non-relativistic version of the DKP equation for spinless particles. We consider the free particle and, as a non-trivial example, we briefly discuss the simple harmonic oscillator.

The formulation of Galilei transformations adopted in this letter involves an embedding in a five-dimensional de Sitter space $\mathcal{G}$ and is such that one obtains a covariant form for non-relativistic physics. It was introduced about ten years ago in [1], followed shortly by a beautiful paper [2] where the idea of adding an extra degree of freedom to the Lagrangian was borrowed from [3]. Further developments and applications are given in [4, 5]. A similar geometrical approach can be found in [6]. An advantage of all this formalism is that Galilean covariance is manifest throughout and quite similar to the relativistic formulation. Also, the problems are often more elegant and simplified, particularly because one considers the *vector* representations of the Galilei group rather than *projective* representations. A simple but elegant argument for the extended space–time is that the free Lagrangian

$$L = \tfrac{1}{2}m\dot{\boldsymbol{x}}^2 \tag{1}$$

although not invariant under Galilei transformations

$$\begin{aligned} \boldsymbol{x} &\to \boldsymbol{x}' = R\boldsymbol{x} - \boldsymbol{v}t + \boldsymbol{a} \\ t &\to t' = t + b \end{aligned} \tag{2}$$

becomes invariant if we extend it as

$$L \to L - m\dot{s} \tag{3}$$

given that s transforms as

$$s \to s - (R\boldsymbol{x}) \cdot \boldsymbol{v} + \tfrac{1}{2}\boldsymbol{v}^2 t + \text{const}. \tag{4}$$

The covariant approach to these Galilei transformations consists of embedding the usual three-dimensional space $\mathcal{E}$ (wherein time t appears as an external parameter) in a five-dimensional de Sitter space $\mathcal{G}$ with coordinates (x, y, z, t, s) (in [4] the fifth coordinate s is identified with $\frac{\boldsymbol{x}^2}{2t}$). As shown in [1], the transformations (2) and (4) leave invariant the scalar product, with metric

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{5}$$

Throughout this letter, the Greek indices, $\mu, \nu, \ldots$, run from 1 to 5 whereas the Latin indices, $j, k, \ldots$, run only from 1 to 3. Also, we denote the inner product corresponding to (5) by

$$\begin{aligned}(x|y) &= g_{\mu\nu} x^\mu y^\nu \\ &= \sum_{i=1}^{n=3} x^i y^i - x^4 y^5 - x^5 y^4. \end{aligned} \tag{6}$$

It is shown in [4] how the *extended* Galilei algebra is obtained as a subalgebra of the Lie algebra associated with the set of linear transformations in $\mathcal{G}$ of type

$$\overline{x}^\mu = G^\mu_\nu x^\nu + a^\mu \tag{7}$$

that leave $(\mathrm{d}x|\mathrm{d}y)$ invariant, and such that $|G| = 1$ with $G^\mu_\nu = \delta^\mu_\nu + \epsilon^\mu_\nu$. This group admits 15 generators of transformations, and the central extension can be seen as the translation generator in the fifth dimension, that is, the mass is a relic of the additional dimension!

From the geometrical point of view, the most natural embedding of $\mathcal{E}$ in $\mathcal{G}$ is given by [4]

$$\boldsymbol{A} \mapsto A = \left(\boldsymbol{A}, A_4, \frac{\boldsymbol{A}^2}{2A_4}\right) \qquad \text{with} \quad \boldsymbol{A} = (A^1, A^2, A^3) \in \mathcal{E} \qquad A \in \mathcal{G}. \tag{8}$$

The purpose of this letter is to apply this formalism to construct non-relativistic wave equations following the existing DKP approach, valid for relativistic wave equations for arbitrary spins [7, 8]. Hereafter we restrict our study to the spin-zero particles; we plan to investigate higher-spin representations in another paper. In ordinary (3 + 1)-dimensional space–time, the DKP equations are a particular case of equations of the form (see the beautiful review by Krajcik and Nieto [7] and the references therein)

$$(\mathrm{i}\alpha_\mu p^\mu + k)\psi = 0 \tag{9}$$

where the α_μ are given by representations of $so(5)$:

$$\alpha_\mu \equiv J_{\mu 5}, \qquad J_{\mu\nu} = -\mathrm{i}[\alpha_\mu, \alpha_\nu], \qquad J_{55} = 0. \tag{10}$$

Similarly, in $d + 1$ dimensions, one works with representations of the Lie algebra $so(d + 2)$. In particular, this algebra is $so(4)$ (isomorphic to $su(2) \oplus su(2)$, which is *not* simple) in 2 + 1 dimensions, and $so(3)$ in 1+1 dimensions. Therefore, in 4+1 dimensions, we will construct the DKP algebra from representations of $so(6)$ (a more complete investigation will be published separately). Just like in 3+1 dimensions, the five- and ten-dimensional representations of $so(5)$ provide the DKP equations for spinless and spin-one particles, respectively, in 4+1 dimensions, the spin-zero particles are described by the fundamental representation of dimension six. Hereafter our construction is therefore based on this representation.

Before considering the DKP equation let us illustrate the Galilei covariant formalism to derive the Schrödinger equation. Using the first Casimir invariant P^2 of the 'inhomogeneous Poincaré' algebra associated with (7), one can write

$$\partial_\mu \partial^\mu \Psi = k^2 \Psi \qquad \partial_5 \Psi = -\mathrm{i} m \Psi \tag{11}$$

where k^2 is a positive constant, $\Psi(x)$ is a vector in the Hilbert space with $\Psi(x) = \langle x|\Psi\rangle$, and m is a parameter defining the value of the invariant P_5 in the representation, according to Schur's lemmas. Using the Galilean embedding (8), the first of equations (11) is

$$(\nabla^2 - 2\partial_4 \partial_5)\Psi = 0 \tag{12}$$

where $x^4 = t$. Using the second of equations (11), we have

$$\mathrm{i}\partial_t \Psi = -\frac{1}{2m}\nabla^2 \Psi. \tag{13}$$

This is the Schrödinger equation for a free particule of mass m.

Now let us at last turn our attention to the DKP equation, given by

$$(\beta^\mu \partial_\mu + k)\psi = 0 \tag{14}$$

where k is an arbitrary constant that reflects the fact that the total energy is defined up to a constant, as we shall see later. The *five* matrices β satisfy the DKP algebra

$$\beta^\mu \beta^\nu \beta^\lambda + \beta^\lambda \beta^\nu \beta^\mu = g^{\mu\nu}\beta^\lambda + g^{\lambda\nu}\beta^\mu \tag{15}$$

where $g^{\mu\nu}$ is now the Galilean metric (5). (In the literature, where the purpose is usually to construct *relativistic* wave equations, $g^{\mu\nu}$ is just the four-dimensional Lorentz metric.) The dimension of the matrices β depend on the spin. Here, we construct *non-relativistic* wave equations for spinless particles, using the metric space associated with (5). Therefore we need five matrices, each of dimension six. We shall work with the following representation:

$$\beta^1 = \begin{pmatrix} 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 1&0&0&0&0&0 \end{pmatrix} \qquad \beta^2 = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&1&0&0&0&0 \end{pmatrix}$$

$$\beta^3 = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&1&0&0&0 \end{pmatrix} \qquad \beta^4 = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&-1&0 \end{pmatrix} \tag{16}$$

$$\beta^5 = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&-1&0&0 \end{pmatrix}.$$

They satisfy the DKP algebra (15) with $g^{\mu\nu}$ given by (5).

The DKP equation (14) is obtained from the Lagrangian

$$\begin{aligned} L_{\mathrm{DKP}} &= \overline{\psi}\left(\tfrac{1}{2}\beta^\mu(\overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu) + k\right)\psi \\ &= \tfrac{1}{2}\overline{\psi}\beta^\mu \partial_\mu \psi - \tfrac{1}{2}(\partial_\mu \overline{\psi})\beta^\mu \psi + k\overline{\psi}\psi. \end{aligned} \tag{17}$$

The adjoint of ψ is given by $\overline{\psi} \equiv \psi^\dagger \eta$, where

$$\eta = (\beta^4 + \beta^5)^2 + \mathbf{1}. \tag{18}$$

One can show that

$$\begin{aligned} \beta^i \eta &= -\eta\beta^i \\ \beta^4\eta &= \eta\beta^5 \\ \beta^5\eta &= \eta\beta^4. \end{aligned} \tag{19}$$

The Euler–Lagrange equations for the adjoint spinor $\overline{\psi}$ give the DKP equation (14), whereas the corresponding equations for the spinor ψ provide the adjoint equation

$$\overline{\psi}(\beta^\mu \overleftarrow{\partial}_\mu - k) = 0. \tag{20}$$

There exists a conserved five-current

$$j^\mu = \overline{\psi}\beta^\mu\psi. \tag{21}$$

Indeed

$$\begin{aligned} \partial_\mu j^\mu &= (\partial_\mu \overline{\psi}\beta^\mu)\psi + \overline{\psi}(\beta^\mu\partial_\mu\psi) \\ &= (k\overline{\psi})\psi + \overline{\psi}(-k\psi) \\ &= 0 \end{aligned} \tag{22}$$

where (14) and (20) have been used. Also, developing the sum,

$$\partial_\mu j^\mu = \partial_k(\overline{\psi}\beta^k\psi) + \partial_t(\overline{\psi}\beta^4\psi) + \partial_5(\overline{\psi}\beta^5\psi) = 0 \tag{23}$$

one can make the identifications: $j^k \equiv (\overline{\psi}\beta^k\psi)$, $\rho \equiv (\overline{\psi}\beta^4\psi)$ and $j^5 \equiv (\overline{\psi}\beta^5\psi) = 0$.

The simplest example, the free particle, is considered as follows. First, let us rewrite (14) in a form involving the five-momentum:

$$(\beta^\mu p_\mu + k)\psi = 0 \tag{24}$$

where $p^\mu = (\boldsymbol{p}, m, E)$ and $p_\mu = (\boldsymbol{p}, -E, -m)$. Also, $p_\mu = -\mathrm{i}\hbar\partial_\mu = (-\mathrm{i}\hbar\nabla, -\mathrm{i}\hbar\partial_t, -\mathrm{i}\hbar\partial_5)$, and the factor $-\mathrm{i}\hbar$ is absorbed in the k of equation (24). Written more explicitly, (24) becomes

$$\begin{aligned} &(\boldsymbol{\beta}\cdot\boldsymbol{p} - \beta^4 E - m\beta^5 + k)\psi = 0 \\ &(\boldsymbol{\beta}\cdot\boldsymbol{p} - m\beta^5 + k)\psi = \mathrm{i}\hbar\beta^4\partial_t\psi \end{aligned} \tag{25}$$

working with the stationary states, for which $E\psi = \mathrm{i}\hbar\partial_t\psi$.

The matrices (16) describe particles with spin-zero, represented by the 'DKP spinor':

$$\psi = \begin{pmatrix} A_x \\ A_y \\ A_z \\ \theta \\ \varphi \\ \phi \end{pmatrix}. \tag{26}$$

In matrix form (25) is, using (16) and (26),

$$\begin{pmatrix} k & 0 & 0 & 0 & 0 & p_x \\ 0 & k & 0 & 0 & 0 & p_y \\ 0 & 0 & k & 0 & 0 & p_z \\ 0 & 0 & 0 & k & 0 & 0 \\ 0 & 0 & 0 & 0 & k & -m \\ p_x & p_y & p_z & m & 0 & k \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \\ \theta \\ \varphi \\ \phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ E\phi \\ 0 \\ -E\varphi \end{pmatrix} \tag{27}$$

for the stationary states.

Solving (27) for ϕ, and absorbing the constant k into the energy by the redefinition

$$E \to E + \frac{k^2}{2m} \tag{28}$$

one obtains the Schrödinger equation

$$E\phi = \frac{\boldsymbol{p}^2}{2m}\phi. \tag{29}$$

Also, one obtains from (27) that, once the solution ϕ is known, the spinor (26) reduces to

$$\psi_{\mathrm{free}} = \begin{pmatrix} -\boldsymbol{p}/k \\ E/k \\ m/k \\ 1 \end{pmatrix} \phi. \tag{30}$$

As another example, let us consider the 'DKP harmonic oscillator', where, essentially, one performs the substitution [9]

$$\boldsymbol{p} \to \boldsymbol{p} - \mathrm{i}m\omega\eta\boldsymbol{r} \tag{31}$$

into equation (25), which then becomes

$$[\boldsymbol{\beta} \cdot (\boldsymbol{p} - \mathrm{i}m\omega\eta\boldsymbol{r}) - m\beta^5 + k]\psi = \mathrm{i}\hbar\beta^4\partial_t\psi. \tag{32}$$

Using again the 'DKP spinor' (26), the DKP equation can be written, for the stationary states, as

$$\begin{aligned} &k\boldsymbol{A} + (\boldsymbol{p} + \mathrm{i}m\omega\boldsymbol{r})\phi = \boldsymbol{0} \\ &k\theta = E\phi \\ &k\varphi = m\phi \\ &(\boldsymbol{p} - \mathrm{i}m\omega\boldsymbol{r}) \cdot \boldsymbol{A} + m\theta + k\phi = -E\varphi. \end{aligned} \tag{33}$$

Again, by reducing (33) in terms of ϕ only, one obtains

$$-(\boldsymbol{p} - \mathrm{i}m\omega\boldsymbol{r}) \cdot (\boldsymbol{p} + \mathrm{i}m\omega\boldsymbol{r})\phi + 2mE\phi + k^2\phi = 0. \tag{34}$$

This can be further reduced (using definition (28), and not forgetting that $\boldsymbol{p}$ and $\boldsymbol{r}$ satisfy the canonical commutation relations!) to

$$E\phi = \left(\frac{\boldsymbol{p}^2}{2m} + \frac{1}{2}m\omega^2\boldsymbol{r}^2 + \frac{3}{2}\hbar\omega\right)\phi \tag{35}$$

which describes the usual simple harmonic oscillator.

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